ISSN 2229-5518

Fixed fuzzy point theorem in metric spaces and its applications to fuzzy differential equations

J.Jeyachristy Priskillal, P.Thangavelu

Abstract— In this paper, we prove a fixed fuzzy point theorem for fuzzy mapping in a complete metric space and give applications to fuzzy differential equations.

Index Terms— complete metric space, fuzzy set, fuzzy fixed point, fuzzy differential equations, fuzzy mapping, Hausdorff distance, nondecreasing function.

1 Introduction

In mathematics, the Banach fixed point theorem[2] (also known as the contraction mapping theorem or contraction mapping principle) is an important tool in the theory of metric spaces. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points. As an extension of Banach contraction theorem, many Mathematicians have studied the concept of fixed point theorem in metric spaces and its application to differential equations[5,10,11]. In 2003, Lakshmikantham.V and Mohapatra[12] gave application of the Banach contraction theorem to fuzzy differential equations. Heilpern[6], introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings in complete metric spaces. Continuing this, fixed point theorem for fuzzy mappings in complete metric spaces has been studied by many authors[1,3,4,13,14,15]. Hemant Kumar Nashine[7] et al. gave a fixed point theorem for fuzzy mapping over a complete metric space and gave application to fuzzy differential equations[8,9]. In this paper, we prove a fixed point theorem for fuzzy mapping in a complete metric space and give applications to fuzzy differential equations. We need the following for main result.

Definition 1.1:

Let A be a fuzzy set in X. If $\alpha\in[0,1]$, then the α-*level* set A_α of A is defined as

$A_\alpha=\{x: A(x)\geq\alpha\}$.

Definition 1.1:

For $\alpha \in (0,1]$, the fuzzy point x_α of X is the fuzzy set of X given by $x_\alpha(x) = \alpha$ and $x_\alpha(z) = 0$ if $z \neq x$.

- *J.Jeyachristy Priskillal, India. E-mail: jeyachristypriskillal@gmail.com*
- *P.Thangavelu, India. E-mail: ptvelu12@gmail.com*

Definition 1.3:

Let $W(X)$ be a collection of approximation quantities. The family $W_\alpha(X) = \{A\in I^X : A_\alpha$ is nonempty, compact and convex$\}$. Let p_α be a α-space, D_α be a α-distance, D be the distance and H be the Hausdorff distance. Let $A,B\in W(X)$

$$p_\alpha(A,B) = \inf_{x\in A_\alpha,\, y\in B_\alpha} d(x,y),$$

$$D_\alpha(A,B) = H(A_\alpha,B_\alpha),$$

$$D(A,B) = \sup_\alpha D_\alpha(A,B).$$

Definition 1.4 :

Let X be an arbitrary set, Y be any metric linear space and I = [0,1]. A fuzzy set of X is an element of I^X. F is called *fuzzy mapping* iff F is a mapping from the set X into a family $W(X)\subset I^X$, that is $F(x)\in W(X)$ for each $x\in X$.

Lemma 1.5:

Let (X,d) be a metric space. Let $A,B\in W(X)$ and $x,y\in X$.

1. $x_\alpha\subset A$ if $p_\alpha(x,A)=0$,
2. $p_\alpha(x,A)\leq d(x,y)+p_\alpha(y,A)$,
3. If $x_\alpha\subset A$, then $p_\alpha(x,A)\leq D_\alpha(A,B)$.

Definition 1.6:

A fuzzy point x_α in X is called a *fixed fuzzy point* of a fuzzy mapping F if $x_\alpha\subset F(x)$, that is, the fixed degree of x in $F(x)$ is atleast α. If $x_1\subset F(x)$, then x is a fixed point of fuzzy mapping F.

2 Fixed fuzzy point theorem in metric space

Theorem 2.1:

Let $\alpha\in(0,1]$, (X,d) be a complete metric space and F be a fuzzy mapping from X onto $W(X)$ such that there exists a nondecreasing function $K:[0,\infty)\to[0,\infty)$ satisfying $\sum_{n=1}^{\infty} K^n(t) < \infty$, $\forall t > 0$, $K(0)=0$, $D_\alpha(F(x),F(y))\le K(d(x,y))$ for all $x,y\in X$ and K is continuous at the origin. Then x_α is a fixed fuzzy point of F.

Proof:

Let $x_0 \in X$ and $F : X \to W_\alpha(X)$ be a fuzzy mapping. Suppose there exists $x_1 \in (F(x_0))_\alpha$ such that $K : [0,\infty) \to [0,\infty)$ satisfying $\sum_{n=1}^{\infty} K^n(t) < \infty$, . Since $F(x_1))_\alpha$ is nonempty compact subset of X, then there exists $x_2 \in (F(x_1))_\alpha$ such that by lemma 1.5(3) and by our hypothesis,

$$\begin{aligned} d(x_1,x_2) &= p_\alpha(x_1,F(x_1)) \\ &\le D_\alpha(F(x_0),F(x_1)) \\ &\le K(d(x_0,x_1)). \end{aligned}$$

By induction, construct a sequence $\{x_n\}$ in X such that $x_n\in(F(x_{n-1}))_\alpha$, by lemma 1.5(3) and by our hypothesis,

$$\begin{aligned} d(x_n, x_{n-1}) &= p_\alpha(x_n , F(x_n)) \\ &\le D_\alpha(F(x_{n-1}), F(x_n)) \\ &\le K(d(x_{n-1}, x_n)) \\ &= K(p_\alpha(x_{n-1}, F(x_{n-1}))) \\ &\le K(D_\alpha(F(x_{n-2}), F(x_{n-1}))) \\ &\le K(K(d(x_{n-2}, x_{n-1}))) \\ &\vdots \\ &\le K^n(d(x_0,x_1)). \end{aligned}$$

Continuing this process, we can get,

$$\begin{aligned} d(x_n,x_m) &\le d(x_n,x_{n+1})+ \dots +d(x_{n+m-1},x_{n+m}) \\ &\le K^n(d(x_0,x_1))+\dots + K^{n+m-1}(d(x_0,x_1)) \\ &= \sum_{i=n}^{n+m-1} K^i(d(x_0,x_1)). \end{aligned}$$

Since $\sum_{n=1}^{\infty} K^n(t) < \infty$, $\{x_n\}$ is a Cauchy sequence in X. Since (X,d) is a complete metric space, there exists $x\in X$, such that $(d(x_n, x))\to 0$. Now by lemma 1.5(2,3) and K is continuous at the origin,

$$\begin{aligned} p_\alpha(x,F(x)) &\le d(x,x_n)+ p_\alpha(x_n,F(x)) \\ &\le d(x,x_n)+ D_\alpha(F(x_{n-1}),F(x)) \\ &\le d(x,x_n)+K(d(x_{n-1},x)) \\ &\to 0+0 = 0. \end{aligned}$$

Therefore, $p_\alpha(x,F(x)) =0$ and by lemma 1.5(1), $x_\alpha \subset F(x)$.

Example 2.2:

Let $X =[0,1]$, $d : X \times X \to X$ be the Euclidean metric and $\alpha \in (0,\frac{1}{2})$. The fuzzy mapping $F : X \to I^X$ is defined by

$$F(0)(x)=\begin{cases} 1, & x = 0, \\ \alpha, & x \in (0,\frac{1}{2}], \\ \frac{\alpha}{2} & x \in (\frac{1}{2},1], \end{cases}$$

$$F(1)(x)=\begin{cases} 1, & x = 0, \\ 2\alpha, & x \in (0,\frac{1}{2}], \\ \frac{\alpha}{2} & x \in (\frac{1}{2},1], \end{cases}$$

and for $z\in(0,1)$, $$F(z)(x)=\begin{cases} 1, & x = 0, \\ \alpha, & x \in (0,\frac{1}{2}], \\ 0 & x \in (\frac{1}{2},1], \end{cases}$$

Then $F(0)_1=F(z)_1=F(1)_1=\{0\}$, $F(0)_\alpha= F(z)_\alpha= F(1)_\alpha=[0,\frac{1}{2}]$, and

$F(0)_{\frac{\alpha}{2}} = F(1)_{\frac{\alpha}{2}} = [0,1]$, $F(z)_{\frac{\alpha}{2}} = [0,\frac{1}{2}]$.

Now,

$D_1(F(x),F(y)) = H(F(x)_1,F(y)_1)=0$ for all $x,y\in X$,

$D_\alpha(F(x),F(y)) = H(F(x)_\alpha, F(y)_\alpha)=0$ for all $x,y\in X$,

$D_{\frac{\alpha}{2}}(F(x),F(y)) = H(F(x)_{\frac{\alpha}{2}}, F(y)_{\frac{\alpha}{2}})=0$ for all $x,y\in\{0,1\}$ and for all $x,y\in(0,1)$,

$D_{\frac{\alpha}{2}}(F(x),F(y))= H(F(x)_{\frac{\alpha}{2}}, F(y)_{\frac{\alpha}{2}})=\frac{1}{2}$ for all $x\in\{0,1\}$ and for all $y\in(0,1)$.

Define a function $K:[0,\infty)\to[0,\infty)$ by $K(t)=\frac{t}{t+1}$.

Clearly, $\sum_{n=1}^{\infty} K^n(t) < \infty$, $K(0)=0$.

For all $x,y\in X$, $D_\alpha(F(x),F(y)) = 0\le K(d(x,y))$.

The hypothesis is verified.

Then 0 is the fixed fuzzy point.

The theorem is justified.

3 Applications to Fuzzy Differential Equations

Consider the boundary value problem

$$x''(t) = f(t, x(t), x'(t)), t \in J = [a,b],$$
$$x(t_1) = x_1, x(t_2) = x_2, t_1, t_2 \in J,$$

where $f : J \times E^n \times E^n \to E^n$ is a continuous function. This problem is equivalent to the integral equation

$$x(t) = \int_{t_1}^{t_2} G(t,s) f(s, x(s), x'(s)) ds + \beta(t).$$

where Green's function G is given by

$$G(t,s) = \begin{cases} \dfrac{(t_2 - t)(s - t_1)}{(t_2 - t_1)}, & t_1 \le s \le t \le t_2, \\ \dfrac{(t_2 - s)(t - t_1)}{(t_2 - t_1)} & t_1 \le t \le s \le t_2, \end{cases}$$

and $\beta(t)$ satisfies $\beta'' = 0, \beta(t_1) = x_1, \beta(t_2) = x_2$. Let us recall some properties of $G(t,s)$ namely,

$$\int_{t_1}^{t_2} |G(t,s)| ds \le \frac{(t_2 - t_1)^2}{8}$$

and

$$\int_{t_1}^{t_2} |G_t(t,s)| ds \le \frac{(t_2 - t_1)}{2}.$$

Now, we shall prove the existence of the result for the above boundary value problem by using our theorem 2.1.

***Theorem 3.1*:**

Let $f : J \times E^n \times E^n \to E^n$ and there exists $\gamma > 0, \delta > 0$ such that

$$|f(t, x(t), x'(t)) - f(t, y(t), y'(t))| \le \gamma |x(t) - y(t)| + \delta |x'(t) - y'(t)|$$

for all $(t, x, x'), (t, y, y') \in J \times E^n \times E^n$ and $\gamma \le \delta$. Then the boundary value problem has a solution.

Proof:

Consider $C = C^1[[t_1,t_2],W(X)]$ with the metric

$$D(x,y) = \max_{t_1 \le t \le t_2} [\gamma |x(t) - y(t)| + \delta |x'(t) - y'(t)|].$$

The space (C,D) is a complete metric space. Define the operator $F{:}C \to C$ by

$$Fx(t) = \int_{t_1}^{t_2} G(t,s) f(s, x(s), x'(s)) ds + \beta(t).$$

Define $K{:}[0,\infty) \to [0,\infty)$ by $K(t)=2\delta t$. Then using the properties of the metric d, we get successively,

$$|Fx(t) - Fy(t)| \le \int_{t_1}^{t_2} |G(t,s)|$$
$$|f(s, x(s), x'(s)) - f(s, y(s), y'(s))| ds$$
$$\le D(x,y) \int_{t_1}^{t_2} |G(t,s)| ds$$
$$\le D(x,y) \frac{(t_2 - t_1)^2}{8}$$
$$\le \frac{D(x,y)}{8}$$
$$\le D(x,y)$$

and

$$|(Fx)'(t) - (Fy)'(t)| \le \int_{t_1}^{t_2} |G_t(t,s)|$$
$$|f(s, x(s), x'(s)) - f(s, y(s), y'(s))| ds$$
$$\le D(x,y) \int_{t_1}^{t_2} |G_t(t,s)| ds$$
$$\le D(x,y) \frac{(t_2 - t_1)}{2}$$
$$\le D(x,y).$$

Now, we have

$$D[Fx, Fy] \le \gamma D(x,y) + \delta D(x,y)$$

$\leq 2\delta\ D(x,y)$

$= K(D(x,y))$

and $\sum_{n=1}^{\infty} K^n(t) = \sum_{n=1}^{\infty} (2\delta)^n t < \infty, K(0) = 0.$

We obtain $D(Fx,Fy) \leq K(D(x,y))$.

Therefore, Theorem 2.1 applies to F which has a fixed point $x^* \in C$, that is x^* is a solution of the boundary value problem.

REFERENCES

[1]. Abbas,M, Damjanivic,B, Lazovic,R, "Fuzzy common fixed point theorems for generalized contractive mappings", *Appl. Math. Lett.*, 23(11),1326-1330, 2010.

[2]. Banach,S. Sur les operations dans les ensembles abstraits et leur application aux equations integrals, *Fundam. Math*. 3,133-181, 1922.

[3]. Ciric,LB, Abbas,M, Damjanovic,B, Saadati,R, "Common fuzzy fixed point theorems in ordered metric spaces", *Math. Comp. Model*., 53, 1737-1741, 2011.

[4]. Estruch, VD, Vidal,A, "A note on fixed fuzzy points for fuzzy mappings", *Rend. Ist. Mat. Univ. Trieste*., 32, 39-45, 2001.

[5]. J.Harjani and K.Sadarangani, "Generalized Contractions in Partially Ordered Sets and Applications to Ordinary Differential Equations", *Nonlinear Analysis*, 72, 1188-1197, 2010.

[6]. S.Heilpern, "Fuzzy mappings and fixed point theorem", *J. Math. Anal. Appl.* 83, 566-569, 1981.

[7]. Hemant Kumar Nashine, Calogero Vetro, Wiyada Kumam and Poom Kumam, "Fixed point theorems for fuzzy mappings and applications to ordinary fuzzy differential equations",*Advances in Difference Equations*. 232(1) , 1-14, 2014.

[8]. Jong Yeoul Park and Hyo Keun Han, "Fuzzy differential equations", *Fuzzy Sets and Systems*, 110 , 69–77, 2000.

[9]. Jong Yeoul Park, S.Y.Lee and H.M.Kee, "The existence of solution for fuzzy differential equations with infinite delays", *Indian journal of pure and applied Mathematics*, 31(2), 137-151, 2000.

[10]. Juan J. Nieto and Rosana Rodriguez-Lopez, "Contractive Mapping Theorems in Partially Ordered Sets and Applications to Ordinary Differential Equations", *Order*, 22, 223–239,2005.

[11]. Juan J. Nieto and Rosana Rodriguez-Lopez, "Existence and Uniqueness of Fixed Point in Partially Ordered Sets and Applications to Ordinary Differential Equations", *Acta Mathematica Sinica, English Series*, 23(12), 2205-2212, 2007.

[12]. Lakshmikantham,V,Mohapatra, "Theory of fuzzy differential equations and inclusions", *Taylor and Francis*, London, 2003.

[13]. Rashwan,RA, Ahmed,MA, "Common fixed point theorems for fuzzy mappings", *Arch. Math*., 38, 219-226, 2002.

[14]. Shaban Sedghi, Nabi Shobe and Ishak Altun, Fixed fuzzy point for fuzzy mappings in complete metric spaces. *Mathematical communications*., 13, 289-294, 2008.

[15]. D.Turkoglu and B.E.Rhoades, "A fixed fuzzy point for fuzzy mapping in complete metric spaces", *Mathematical communications*, 10, 115-121, 2005.